Comments on 'Existence of axially symmetric solutions in SU(2)-Yang Mills and related theories [hep-th/9907222]'

Burkhard Kleihaus
NUIM, Department of Mathematical Physics, Maynooth, Co. Kildare, Ireland

and

Jutta Kunz
Fachbereich Physik, Universität Oldenburg, D-26111 Oldenburg, Germany
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In [hep-th/9907222] Hannibal claims to exclude the existence of particle-like static axially symmetric non-abelian solutions in $SU(2)$ Einstein-Yang-Mills-dilaton theory. His argument is based on the asymptotic behaviour of such solutions. Here we disprove his claim by giving explicitly the asymptotic form of non-abelian solutions with winding number $n = 2$.

Particle-like static axially symmetric solutions of Yang-Mills-dilaton (YMD) [1] and Einstein-Yang-Mills-dilaton (EYMD) theory [2,3] have been investigated numerically and analytically [4] in recent years. The gauge potential, given in a singular gauge in [1–3], can locally be gauge transformed into a regular form [4]. On intersecting neighbourhoods the regular gauge potentials can be gauge transformed into each other by regular gauge transformations. Hence these solutions are globally regular.

Recently, Hannibal [7] claimed to have shown that the static axially symmetric solutions discussed in [1–3,5,6,4] are singular in the gauge field part. However, in [7] he only observed that the *gauge potential* of refs. [1–3,5,6] does not obey a set of *sufficient* regularity conditions [8]. Repeating his claim in [9], and noting that the "singular" solutions can be locally gauge transformed into regular solutions as discussed in [4,10], Hannibal apparently uses the equality *solution* = *gauge potential*, despite the fact that the gauge potential is not uniquely defined and the *same* solution can be given in many *different* gauges.

In [9] Hannibal then turns to the question of existence of static axially symmetric solutions of YMD and EYMD theory. He presents as his results, that i) "there exist only embedded abelian particle-like solutions" and ii) "the solutions constructed by Kleihaus and Kunz are shown to be gauge equivalent to these".

Particle-like solutions of EYMD and related theories − which are not embedded abelian solutions − have been investigated in the past decade by several authors (see e. g. [11]) and it is generally accepted that these solutions are genuine non-abelian solutions. For rigorous proofs of the existence of these solutions see e. g. [12].

Concerning the static axially symmetric solutions, the argument of Hannibal relies on the fact that he was not able to find asymptotic solutions, which possess the correct symmetries with respect to the discrete transformation

$$\mathrm{S}_{(z\leftrightarrow -z)} : \theta \longrightarrow \pi - \theta \ . \tag{1}$$

However, he chooses a gauge in which this symmetry is lost for all non-trivial gauge field functions parameterizing the gauge potential. Apparently he takes this into account for one of the functions, but does not take it into consideration for the other two functions. Without taking into account the correct symmetry behaviour of the gauge field functions the results are not reliable.

By giving a counterexample, we here disprove Hannibal's claim, *that "there exist only embedded abelian particle-like solutions ..." [9], because asymptotic solutions for genuine non-abelian gauge potentials do not exist.* This counterexample represents the correct asymptotic behaviour for a *non-abelian* static axially symmetric solution of EYMD theory [2,3,5,6].

Using polar coordinates (r, θ, φ), we parameterize the static axially symmetric $su(2)$ gauge potential as [5]

$$A_\mu dx^\mu = \frac{1}{2g}\left\{\left[\frac{H_1}{r}dr + (1-H_2)d\theta\right]\tau_\varphi^n - n\sin\theta\left[H_3\tau_r^n + (1-H_4)\tau_\theta^n\right]d\varphi\right\} \ , \tag{2}$$

where H_i are functions of r and θ and n denotes the winding number. For convenience we set the gauge coupling constant g equal to one in the following. The $su(2)$ matrices $\tau_\varphi^n, \tau_r^n, \tau_\theta^n$ are defined in terms of Pauli matrices τ_1, τ_2, τ_3 by

$$\tau_\varphi^n = -\sin(n\varphi)\tau_1 + \cos(n\varphi)\tau_2 \ , \tau_r^n = \sin\theta\tau_\rho^n + \cos\theta\tau_3 \ , \tau_\theta^n = \cos\theta\tau_\rho^n - \sin\theta\tau_3 \ , \tag{3}$$

with $\tau_\rho^n = \cos(n\varphi)\tau_1 + \sin(n\varphi)\tau_2$.

In order to compare with ref. [9] we parameterize the functions H_i as [7]

$$H_1 = \left[\tilde{F}_1 \sin^2\theta + \left(nf(r) + \tilde{F}_2\right)\right] \cos\theta \sin^{|n|}\theta \ , \tag{4}$$

$$(1 - H_2) = \left[nf(r) + \tilde{F}_1 \sin^2\theta\cos^2\theta - \left(nf(r) + \tilde{F}_2\right)\sin^2\theta\right] \sin^{|n|-1}\theta \ , \tag{5}$$

$$H_3 = \left[\left(f(r) + \tilde{F}_3\sin^2\theta\right)\cos\theta\sin^{|n|-1}\theta\right]\sin\theta + \tilde{F}_4\sin\theta\cos\theta \tag{6}$$

$$= F_3\sin\theta + F_4\cos\theta \ , \tag{7}$$

$$(1 - H_4) = \left[\left(f(r) + \tilde{F}_3\sin^2\theta\right)\cos\theta\sin^{|n|-1}\theta\right]\cos\theta - \tilde{F}_4\sin^2\theta \tag{8}$$

$$= F_3\cos\theta - F_4\sin\theta \ , \tag{9}$$

where $\tilde{F}_i$ are continuous functions of r and θ and $F_3 = (f(r) + \tilde{F}_3\sin^2\theta)\cos\theta\sin^{|n|-1}\theta$, $F_4 = \tilde{F}_4\sin\theta$ have been introduced for later convenience. Assuming that $\tilde{F}_i$ are regular functions of r^2 and $\sin^2\theta$ [8], this parameterization of the ansatz guarantees that the gauge potential is regular on the z-axis ($|z| > 0$) and that the functions H_1 and H_3 are odd under the transformation $\mathrm{S}_{(z\leftrightarrow -z)} : \theta \to \pi - \theta$, whereas the functions H_2 and H_4 are even [7]. In order to guarantee regularity at the origin, additional conditions have to be imposed on the functions $\tilde{F}_i$.

The gauge potential (2) is form invariant under abelian gauge transformations of the form [4]

$$U = \exp\left\{i\Gamma\tau_\varphi^n/2\right\} \ , \tag{10}$$

where Γ is a function of r and θ. The functions H_i transform like

$$H_1 \longrightarrow \hat{H}_1 = H_1 - r\partial_r\Gamma \ , \tag{11}$$

$$H_2 \longrightarrow \hat{H}_2 = H_2 + \partial_\theta\Gamma \ , \tag{12}$$

$$H_3 \longrightarrow \hat{H}_3 = \cos\Gamma(H_3 + \cot\theta) - \sin\Gamma H_4 - \cot\theta \ , \tag{13}$$

$$H_4 \longrightarrow \hat{H}_4 = \sin\Gamma(H_3 + \cot\theta) + \cos\Gamma H_4 \ . \tag{14}$$

Following [9], we now fix the gauge such that $\hat{H}_2 = 1$ and assume that H_2 is given in the form (5). The gauge transformation function is given by

$$\Gamma(r,\theta) = \int_{\theta_0}^{\theta} \{1 - H_2(r,\theta')\}\, d\theta' \ . \tag{15}$$

Again following [9], we choose $\theta_0 = 0$, in order to maintain regularity on the positive z-axis ($z > 0$). Then the gauge potential may diverge on the negative z-axis [9]. Along the (positive) z-axis the function $\hat{H}_1$ is still of the order $O(\sin^{|n|}\theta)$ and may be written locally as $\hat{H}_1 = \bar{F}_1\cos\theta\sin^{|n|}\theta$, with some function $\bar{F}_1(r,\theta)$. The gauge transformed function $\hat{F}_3$ is now of order $O(\sin^{|n|+1}\theta)$ and may be written as $\hat{F}_3 = \bar{F}_3\cos\theta\sin^{|n|+1}\theta$ near the (positive) z-axes, with some function $\bar{F}_3(r,\theta)$. Hence, we find near the (positive) z-axis

$$\hat{H}_1 = \bar{F}_1\cos\theta\sin^{|n|}\theta \ , \tag{16}$$

$$(1 - \hat{H}_2) = 0 \ , \tag{17}$$

$$\hat{H}_3 = \left[\bar{F}_3\sin^{|n|+1}\theta + \bar{F}_4\right]\sin\theta\cos\theta \ , \tag{18}$$

$$(1 - \hat{H}_4) = \bar{F}_3\cos^2\theta\sin^{|n|+1}\theta - \bar{F}_4\sin^2\theta \ . \tag{19}$$

This form of the gauge field functions $\hat{H}_i$ could have been obtained from Eqs. (4)-(8) by setting $f(r) \equiv 0$ and $\tilde{F}_2 = \cos^2\theta\tilde{F}_1$ [9]. However, under the assumption that the functions $\tilde{F}_i$ are continuous everywhere, this would not have been correct. This can be seen as follows.

The function Γ contains an even part and an odd part with respect to the transformation $\mathrm{S}_{(z\leftrightarrow -z)}$ [9]. As a consequence the function $\hat{H}_1$ is no longer odd with respect to $\mathrm{S}_{(z\leftrightarrow -z)}$ [9] and cannot be written in the form (16) with continuous functions $\bar{F}_i$. It is evident from Eqs. (13)-(14) that the functions $\hat{H}_3$ and $\hat{H}_4$ also have lost their antisymmetry, respectively symmetry, with respect to $\mathrm{S}_{(z\leftrightarrow -z)}$. Thus all the functions $\hat{H}_i$ may take finite values on the ρ-axis. If we then insist to use the parameterization Eqs. (16-19) for the gauge transformed functions $\hat{H}_i$, not only near the (positive) z-axis but everywhere, we must allow the functions $\bar{F}_1$ and $\bar{F}_3$ to contain the factor $1/\cos\theta$.

Parameterizing the metric [2,3,5,6,9] by

$$ds^2 = -\mathrm{g}(r,\theta)dt^2 + \frac{\mathrm{m}(r,\theta)}{\mathrm{g}(r,\theta)}(dr^2 + r^2 d\theta^2) + \frac{\mathrm{l}(r,\theta)}{\mathrm{g}(r,\theta)} r^2 \sin^2\theta d\varphi^2 \ , \tag{20}$$

we substitute the gauge transformed ansatz Eq. (2) together with (16)-(19) and (20) into the coupled Einstein, dilaton and Yang-Mills equations and find for the asymptotic EYMD solution with winding number $n = 2$,

$$\bar{F}_1 = -\frac{\mathrm{C}_2}{r^2}\frac{1-\cos\theta}{\cos\theta\sin^2\theta} + (\ldots) \ , \tag{21}$$

$$\bar{F}_3 = \frac{\mathrm{C}_2}{r^2}\frac{2(1-\cos\theta) - \cos\theta\sin^2\theta}{4\cos\theta\sin^4\theta} + (\ldots) \ , \tag{22}$$

$$\bar{F}_4 = \frac{\mathrm{C}_1}{r} + \frac{\mathrm{C}_1}{4r^2}(\bar{\mathrm{g}}_1 - 2\phi_1) + (\ldots) \ , \tag{23}$$

for the gauge field functions and

$$\Phi = \frac{\phi_1}{r} + (\ldots) \ , \quad \mathrm{g} = 1 + \frac{\bar{\mathrm{g}}_1}{r} + \frac{\bar{\mathrm{g}}_1^2}{2r^2} + (\ldots) \ , \quad \mathrm{l} = 1 + \frac{\bar{\mathrm{l}}_2}{r^2} + (\ldots) \ , \quad \mathrm{m} = 1 + \frac{\bar{\mathrm{l}}_2 + \bar{\mathrm{m}}_2\sin^2\theta}{r^2} + (\ldots) \ , \tag{24}$$

for the dilaton function [13] and the metric functions, where C_1, C_2, ϕ_1, $\bar{\mathrm{g}}_1$, $\bar{\mathrm{l}}_2$, $\bar{\mathrm{m}}_2$ are constants [14]. The $(\ldots)$ indicate terms vanishing faster than $1/r^2$, which can not necessarily be expanded in powers of $1/r$. It is easy to see that $\bar{F}_1$ and $\bar{F}_3$ are finite on the positive z-axis and can be expressed as polynomials in $\sin^2\theta$ in the vicinity of the z-axis. Hence, the asymptotic solution is regular on the positive z-axis. On the ρ-axis the gauge field functions $\hat{H}_i$ are finite, as expected. Furthermore, $\hat{H}_1$ can be decomposed into an odd part and an even part with respect $\mathrm{S}_{(z\leftrightarrow -z)}$, where the latter is a function of r only [9]. As a consequence all derivatives $\dfrac{\partial^{2k}\hat{H}_1}{\partial\theta^{2k}}$ vanish on the ρ-axis. On the z-axis the functions $f_{10}(r) := \bar{F}_1(r,\theta=0)$ and $f_{40}(r) := \bar{F}_4(r,\theta=0)$ are not necessarily zero. Hence, this solution is not an embedded abelian solution and has to be classified as type III according to ref. [9]. For winding number $n = 3, 4$ we have constructed asymptotic solutions, too, which possess all the correct symmetry and regularity properties. These simple solutions disprove the claim of Hannibal [9] that only for embedded abelian gauge potentials asymptotic solutions with the correct symmetries can exist in EYMD and related theories.

We have also analysed the solution with winding number $n = 2$ in the Coulomb gauge $rH_{1,r} - H_{2,\theta} = 0$ [15], which was employed in the numerical construction of the solutions [1–3,5,6]. In this gauge the gauge potential is not well defined on the z-axis. However, transforming the asymptotic solution obtained in the Coulomb gauge into the gauge $H_2 = 1$, we find the same form Eqs. (21-23). This shows that at infinity the gauge potential in the Coulomb gauge can be locally gauge transformed into a regular gauge potential and that the asymptotic gauge potentials in the different gauges correspond to the same asymptotic solution. Note, that in the Coulomb gauge the functions H_1 and H_3 are antisymmetric, whereas the functions H_2 and H_4 are symmetric with respect to $\mathrm{S}_{(z\leftrightarrow -z)}$.

In Ref. [9] Hannibal failed to obtain asymptotic non-abelian solutions such as the solution Eqs. (21)-(24). Clearly, he rejected solutions because the functions $\hat{H}_i$ did not possess the symmetry property he erroneously expected. Furthermore, he tried to find the solutions in form of power series in $\sin\theta$. However, the solution Eqs. (21)-(22) contains the function $1/\cos\theta$, which, considered as a power series in $\sin\theta$, has radius of convergence $|\sin\theta| < 1$. Thus, the results at $\theta = \pi/2$ may be not reliable. Unfortunately, the presentation in [9] is not consistent at various places, and it remains unclear which calculations exactly were carried out and whether additional sources of error appeared in the analysis.

We close with some additional comments on ref. [9]:

It is assumed in [9] that the asymptotic solutions can be obtained as a power series in $1/r$. However, if one wants to exclude the existence of all solutions (except embedded abelian solutions), then a proof that all solutions are analytic in the variable $1/r$ at $1/r = 0$ would be necessary. Such a proof is not given in ref. [9]. Indeed, we find that in the Coulomb gauge terms in higher order arise, which can not be expanded in a power series in $1/r$ [15].

It is claimed in [9] that there exists a regular, globally defined gauge potential for the solutions of ref. [1–3,5,6]. However, the "proof" of existence of such a gauge potential is not complete, because no attention is paid to the regularity of the gauge potential at the origin [4]. It can be seen easily, that the gauge potential is not twice differentiable at the origin, if high powers in $\sin\theta$ arise, which are not multiplied by sufficiently high powers in r.

In [9] the static axially symmetric solutions of ref. [1–3,5,6] are classified as gauge equivalent to type II solutions. This is not correct. Defining type II solutions by $f_{10}(r) = \hat{H}_{1,\theta\theta} = 0$ on the z-axis (for $n = 2$) [9], Hannibal then argues, that $f_{10}(r)$ specifies a boundary condition, which can be chosen freely, and that therefore a non-zero $\hat{H}_{1,\theta\theta}$ on the z-axis can not be generated. However, only for a local solution can the function $f_{10}(r)$ be chosen freely, i. e. for any function $f_{10}(r)$ there exists a solution of the differential equations which is valid only in a region near the z-axis. For a global solution the function $f_{10}(r)$ has to be chosen such that the solution fulfills the correct boundary condition at the ρ-axis. If the global solution is unique, the function $f_{10}(r)$ is fixed (provided the gauge is fixed).

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